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Note on the Counter-Pedal Surface of an Ellipsoid.

By Thomas Craig, *Johns Hopkins University.*

In Vol. IV of this Journal I defined the counter-pedal surface of a given surface, and worked out some of its properties for the case of the ellipsoid. The definition given in the paper referred to is: (for the ellipsoid) the counter-pedal surface is the locus of the intersections of the normals to the ellipsoid with the diametral planes parallel to the tangent planes at each point. This surface may also be defined as follows: the counter-pedal surface, when the origin is taken as the pole, is the locus of the intersections of the normals at corresponding points of the ellipsoid and its first pedal.

The ellipsoid being given by

$$\frac{\xi'^2}{a^2} + \frac{\eta'^2}{b^2} + \frac{\zeta'^2}{c^2} = 1,$$

the first pedal is

$$\phi \equiv (a^2x'^2 + b^2y'^2 + c^2z'^2) - (x'^2 + y'^2 + z'^2)^2 = 0.$$

Denoting by P the central perpendicular upon the tangent plane to the ellipsoid, we have for the coördinates x', y', z' of the point corresponding to ξ', η', ζ',

$$x',\ y',\ z' = \frac{P^2\xi'}{a^2},\quad \frac{P^2\eta'}{b^2},\quad \frac{P^2\zeta'}{c^2}.$$

The direction-cosines of the normal to the ellipsoid at a given point ξ', η', ζ' are

$$\frac{P^2\xi'}{a^2},\quad \frac{P^2\eta'}{b^2},\quad \frac{P^2\zeta'}{c^2};$$

the direction-cosines of the normal at the corresponding point x', y', z' on the pedal are proportional to

$$\frac{d\varphi}{dx},\quad \frac{d\varphi}{dy},\quad \frac{d\varphi}{dz}$$

$$= 2x'(a^2 - 2r^2),\ 2y'(b^2 - 2r^2),\ 2z'(c^2 - 2r^2),$$

where $r^2 = x'^2 + y'^2 + z'^2$. Substituting for x', y', z' the above values and we have as the equations of the two normals,

$$\left.\begin{aligned} x &= \frac{c^2\xi'}{a^2\zeta'}\,z - \frac{\beta\xi'}{a^2} \\ y &= \frac{c^2\eta'}{b^2\zeta'}\,z + \frac{\alpha\eta'}{b^2} \end{aligned}\right\}\text{ Ellipsoid.}$$

$$\left.\begin{aligned} x &= \frac{c^2\xi'}{a^2\zeta'}\cdot\frac{a^2-2P^2}{c^2-2P^2}\,z + \frac{P^2\xi'}{a^2}\cdot\frac{a^2-2P^2}{c^2-2P^2} \\ y &= \frac{c^2\eta'}{b^2\zeta'}\cdot\frac{b^2-2P^2}{c^2-2P^2} - \frac{P^2\eta'}{b^2}\cdot\frac{\alpha}{c^2-2P^2} \end{aligned}\right\}\text{ Pedal.}$$

In these x, y, z are the current coördinates of points in the normals, and α, β, $\gamma = b^2 - c^2$, $c^2 - a^2$, $a^2 - b^2$. The values of z obtained from the first and third of these equations, and from the second and fourth, are easily seen to be the same, and therefore the normals at corresponding points on the ellipsoid and its first pedal will intersect. The coördinates of the points of intersection are readily found to be

$$x = \xi'\left(1 - \frac{P^2}{a^2}\right),$$

$$y = \eta'\left(1 - \frac{P^2}{b^2}\right),$$

$$z = \zeta'\left(1 - \frac{P^2}{c^2}\right);$$

but these are the coördinates of a point on the counter-pedal surface, and therefore this latter is the locus of the intersections of normals at corresponding points on the ellipsoid and its first pedal.

This, of course, affords a construction for the normals and tangent planes to the first pedal, two points of the normal being given by the above equations.

The general definition of the counter-pedal, to any surface, as given in the above-mentioned paper, was that it is the locus of the points of intersection of the normals to a surface with planes through a fixed point (the pole) parallel to the tangent planes to the given surface. For convenience the pole may be taken at the origin. It is easy to show now that the above method of generating the counter-pedal in the case of the ellipsoid is applicable to the cases of any surface. A very simple geometrical proof of this is as follows.

Denote by O the pole, and by SQS' the given surface. PQ is the tangent plane at the point Q, and OP the perpendicular upon the tangent plane. Now it is known (*vide* Salmon's Geom. of Three Dimen.) that the sphere described upon OQ as a diameter touches the locus of P, *i. e.* touches the pedal; then if M is the middle point of OQ, the line PM is the normal to the pedal of the surface SS'; but QN the normal to SS' is parallel to OP, and therefore the normal PM passes through the point N; but N is the point on the counter-pedal corresponding to Q on the given surface, hence it follows that the counter-pedal surface to any given surface is the locus of the intersections of the normals at corresponding points on the given surface and its first pedal.

There is another class of surfaces of some interest, which may be called counter-centro-surfaces; these are found by drawing lines through the pole parallel to the normals at each point of a given surface, and laying off upon these lines lengths equal to the radii of curvature at the corresponding points on the surface. The investigation of these surfaces is reserved for the present. It may be remarked, however, that the equation of the counter-centro-surface of the ellipsoid is

$$\frac{a^2x^2}{Q-a^2}+\frac{b^2y^2}{Q-b^2}+\frac{c^2z^2}{Q-c^2}=0,$$

where

$$Q=(a^2x^2+b^2y^2+c^2z^2)^{\frac{1}{2}}.$$

The equation when rationalized is of the eighth degree; the surface has a conjugate quadruple point at the origin, and its principal sections each consist of a conic and a sextic, viz. an ellipse and what may appropriately be called the counter-evolute to the corresponding principal section of the ellipsoid.

It is of course clear that we might have spoken of a quadric in general instead of confining attention merely to the ellipsoid.